Lecture 7. Operational Semantics of F^b

Fb Syntax & Semantics

(Variable) $x ::=$
$[a-z][a-zA-Z0.9\_']$ *

Expr

$e ::=$ | 0 | 1 | -1 | 2 | -2 | ... ⟵

IntPlus of e * e

| $True$ | $False$ ⟵ Const Bool   Const Int

| $e + e$ | $e - e$ | $e * e$ ⟵ Int Bin Op

| $e < e$ | $e > e$ | $e = e$ ⟵ Comparators

| e && e | e || e Bool Bin Op

| $- e$ ⟵ Int Una Op $(-)$ 1 ⟹ -1

| $! e$ ⟵ Bool Una Op

ITE of e*e*e

| If e $Then$ e $Else$ e ⟵ ITE

| Let $x := e$ In e

| x

| Fun $x ⟹ e$

| e e
       ↑   ↑
     func   arg

Semantic

BOOL be

$$\boxed{be} \Rightarrow \boxed{\mathbb{B}}$$
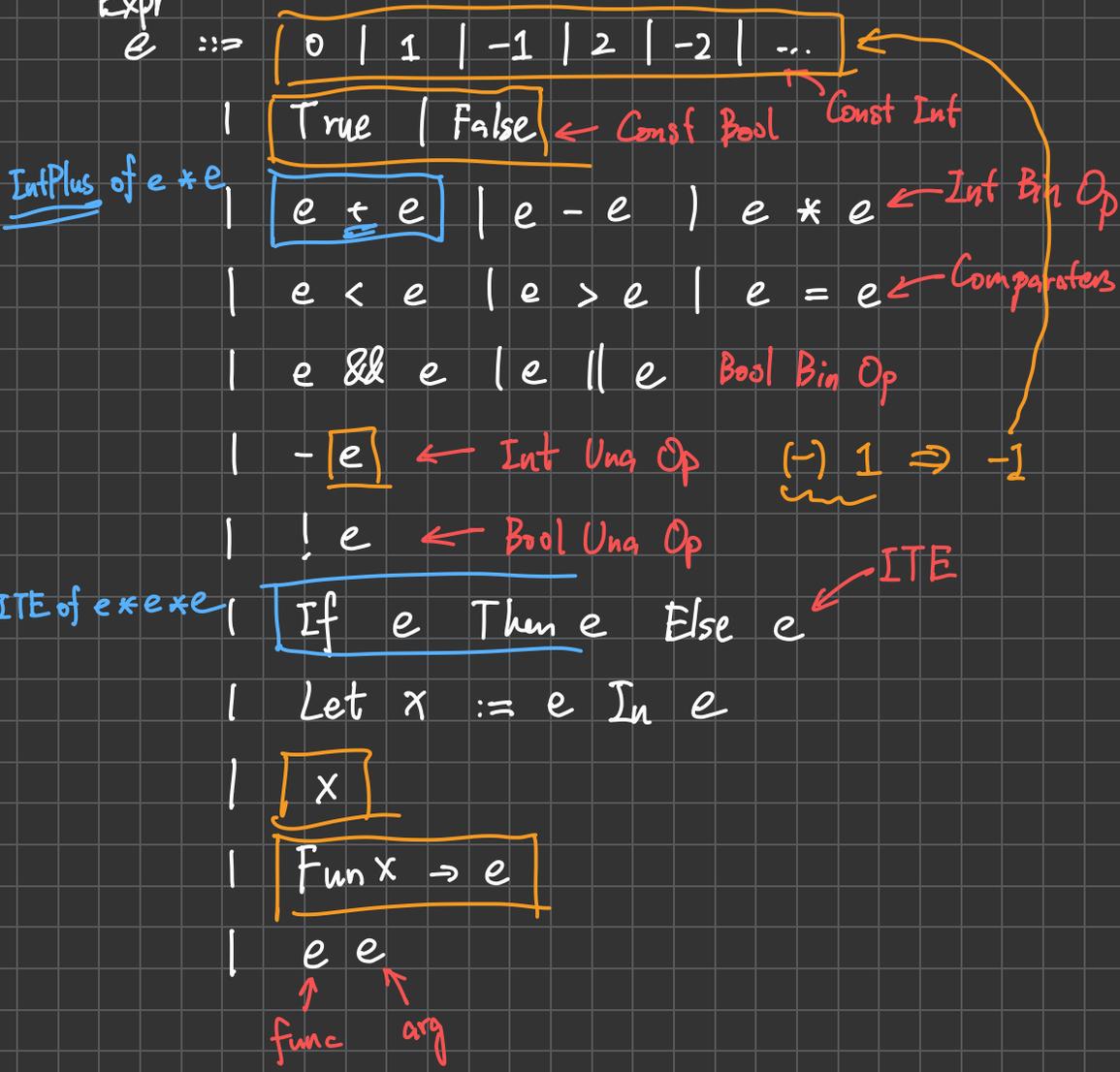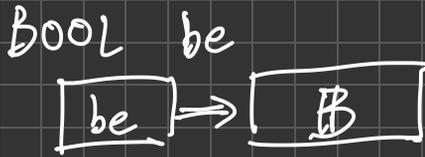
$H_b \quad e$

$$e \Rightarrow e$$

rewrite system

$$(1+2) \Rightarrow 3 \qquad ?$$

$$\text{If true Then 3 Else 4} \Rightarrow 3 \text{ ?}$$

$$(1+2+3) \Rightarrow (3+3)$$

Fb Syntax (Revisited)   (Variable) x

         ↙ Int Constant

                     ← Bool Constant

$\boxed{\text{(Value)}} \quad v ::= x \mid i \mid b \mid \text{Fun } x \Rightarrow e$

$\text{(Expr)} \quad e ::= v \mid e \oplus e \mid \ominus e$ Bin Op   Una Op

$\qquad\qquad\qquad \mid \text{If } e \text{ Then } e \text{ Else } e$

$\qquad\qquad\qquad \mid e \; e$

$\qquad\qquad\qquad \mid \text{Let } x := e \text{ In } e$

Op Sem (Revisited)

$$e \Rightarrow v$$

$$(3 + 1 + 2) \Rightarrow (3 + 3) ? \quad \times$$

$$(\text{If True Then } 3+2 \text{ Else Fun } x \to 1) \Rightarrow 5 \checkmark$$

$$(\text{Fun } x \to x + 1 + 2) \Rightarrow (\text{Fun } x \to x + 3) \times$$

[Variable] $\quad x \Rightarrow x$

[Value] $\overline{v \Rightarrow v}$

[Int Add] $\dfrac{n_1 \in \mathbb{Z} \qquad n_2 \in \mathbb{Z} \\ e_1 \Rightarrow n_1 \qquad e_2 \Rightarrow n_2}{e_1 + e_2 \Rightarrow n_1 + n_2}$

[Bool Eq] $\dfrac{e_1 \Rightarrow b_1 \qquad e_2 \Rightarrow b_2 \qquad b_1, b_2 \in \mathbb{B}}{e_1 = e_2 \Rightarrow b_1 = b_2}$

[Int Eq] $\dfrac{e_1 \Rightarrow n_1 \quad e_2 \Rightarrow n_2 \quad n_1, n_2 \in \mathbb{Z}}{e_1 = e_2 \Rightarrow n_1 = n_2}$

[ITE]
Take1 $\dfrac{\boxed{e_1 \Rightarrow b_1 \quad b_1 \in \mathbb{B}} \quad e_2 \Rightarrow v_2 \quad \boxed{e_3 \Rightarrow v_3}}{\text{If } e_1 \text{ Then } e_2 \text{ Else } e_3 \Rightarrow \begin{cases} v_2 & \text{if } b_1 = \text{True} \\ v_3 & \text{otherwise} \end{cases}}$

$$\nexists \, v \, . \quad \text{If fun } x \to 3 \text{ Then ... Else...} \Rightarrow v$$

$$\text{If True Then } 3 \text{ Else } \boxed{(\text{Fun } x \to 3) + 4}$$

Take 2

[ITE-True] $\dfrac{e_1 \Rightarrow \text{True} \qquad e_2 \Rightarrow v_2}{\text{If } e_1 \text{ Then } e_2 \text{ Else } e_2 \Rightarrow v_2}$

[ITE-False] $\dfrac{e_1 \Rightarrow \text{False} \qquad e_3 \Rightarrow v_3}{\text{If } e_1 \text{ Then } e_2 \text{ Else } e_3 \Rightarrow v_3}$

[Let] $\dfrac{??}{\text{Let } x := e_1 \text{ In } e_2 \Rightarrow ??}$

$$\text{Let } a := 1+2 \text{ In } a + 5$$
$$\downarrow$$
$$\text{Let } a := 3 \text{ In } a + 5$$
$$\downarrow$$
$$3 + 5 \qquad (a+5)[3/a]$$

$$\text{Let } a := 3 \text{ In } (\text{Fun } a \to a + 5) \; 6 \Rightarrow 11$$

Doesn't Matter

Free Variables $\quad FV(e) \cdot$

$FV(a+3) = \{a\}$

$FV(\text{Fun } a \to 3) = \emptyset$

$FV(\text{Fun } a \to a+b) = \{b\}$

$FV: e \to P(x) \quad \leftarrow$ Powerset

$$FV(i) = FV(b) = \emptyset$$
$$FV(x) = \{x\}$$
$$FV(\text{If } e_1 \text{ Then } e_2 \text{ Else } e_3) =$$
$$FV(e_1) \cup FV(e_2) \cup FV(e_3)$$
$$FV(\text{Fun } x \to e) =$$
$$FV(e) \,/\, \{x\}$$

$$e \, [\, v \, / \, x \,]$$

$$e_1 + e_2 \, [\, v \, / \, x \,] = e_1[v/x] + e_2[v/x]$$

$$a + 3 \, [\, 5 \, / \, a \,] = a[5/a] + 3[5/a]$$
$$= 5 + 3$$

$$(\text{Fun } a \to a + b)[\, 5 \, / \, b \,] =$$

$$\left(\text{Fun } x \to e \right)[\, v \, / x \,] = \text{Fun } x \to e$$
$$\left(\text{Fun } x \to e \right)[\, v \, / \, y \,] \text{ where } y \neq x$$
$$= \text{Fun } x \to (e[v/y])$$

$$[\![\text{Let}]\!] \quad \frac{e_1 \Rightarrow v_1 \qquad e_2[v_1/x] \Rightarrow v_2}{\text{Let } x := e_1 \text{ In } e_2 \quad \Rightarrow \quad v_2}$$

Let $x := e_1$ In e_2
with just functions?

$$(\text{fun } x \to e_2)(e_1)$$

$$\text{True} \equiv \text{Fun } tb \to \text{Fun } fb \to tb$$
$$\text{false} \equiv \text{Fun } tb \to \text{Fun } fb \to fb$$
$$\text{ITE} \equiv \text{Fun } c \to \text{Fun } t \to \text{Fun } e \to c\ t\ e$$

If True Then e_1 Else e_2 $\qquad \uparrow$
$\qquad \| \|$
ITE e_1 e_2 e_3 $\qquad\qquad$ Church Encoding

$$(\lambda t . \lambda f . t)\ e_1\ e_2$$
$$\genfrac{}{}{0pt}{}{\}}{\Downarrow}$$
$$(\lambda f . e_1)\ e_2$$
$$\genfrac{}{}{0pt}{}{\}}{\Downarrow}$$
$$e_1$$